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Note

Deterministic soliton automata with a single exterior node*

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Abstract

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Soliton valves have been proposed as molecular switching elements. Their mathematical model is the soliton graph and the soliton automaton introduced in Dassow, Jürgensen (1991). In this paper we continue the study of the logic aspects of soliton switching. There are two cases of special importance: those of deterministic and of strongly deterministic soliton automata. The former have deterministic state transitions in the usual sense of automaton theory. The latter do not only have deterministic state transitions, but also deterministic soliton paths—a much stronger property, as it turns out. In Dassow, Jürgensen (1991) a characterization of indecomposable, strongly deterministic soliton automata was proved and it was shown that their transition monoids are primitive groups of permutations. Roughly speaking, the main difference between deterministic and strongly deterministic soliton automata is that in the former the underlying soliton graphs may contain cycles of odd lengths while such cycles are not permitted in the soliton graphs belonging to strongly deterministic soliton automata. The presence of such cycles renders the analysis quite complicated. In the present paper, we focus on a special class of deterministic soliton automata, that of deterministic soliton automata whose underlying graphs have a single exterior node. Such graphs can be thought of as one kind of basic building blocks in the construction of soliton graphs. We show that these soliton automata have at most two states. This statement is not correct if the assumption of determinism is dropped.

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1. Introduction

In [7] we introduced a formal model of molecular switching based on soliton propagation. Soliton-based molecular switching devices are proposed and discussed in [4, 5, 2, 3], for instance. While many aspects of their realizability are still unclear it still seems important to determine their potential and limitations theoretically.

As a first step, [7] and [6] introduce the formal model, derive some elementary properties, and then focus on one special, but important case, that of strongly deterministic soliton automata. In such a soliton automaton not only the state transitions but also the selection of the soliton paths is deterministic. One of the main results of [7] establishes that the transition monoid of a strongly deterministic soliton automaton is a primitive group of permutations; moreover, a complete characterization of strongly deterministic soliton automata is provided.

A theoretically oriented analysis of the computational power of strongly deterministic soliton automata is provided in [10]. A first step towards a purely algebraic treatment of soliton automata has been achieved in [1].

From a realization point of view strong determinism may be too restrictive a requirement. In a deterministic soliton automaton as opposed to a strongly deterministic one, only the state transitions are deterministic. From a technical point of view this weaker form of determinism may introduce non-determinism of timing.

The characterization of deterministic soliton automata seems to be far more difficult than that of the strongly deterministic ones. So far we can only handle two special cases: In [8] we treat those deterministic soliton automata which have at most one cycle. Again, a description of their transition monoids is obtained. In "most" cases it is again a primitive group of permutations. In the present paper we consider another special case, that of deterministic soliton automata whose underlying soliton graphs have only one exterior node. While this case may seem to have little practical relevance, its analysis can provide some important clues concerning the structural properties of larger soliton graphs; soliton graphs with a single exterior node form one kind of important building blocks in the construction of general soliton graphs. The difficulty of dealing with soliton graphs is illustrated by the fact that not even for this very special case do we obtain a characterization of the deterministic soliton graphs. However, we do obtain a full account of their automaton behavior.

The main result of this paper is as follows: A deterministic soliton automaton whose underlying graph has a single exterior node has at most two states. Hence, its transition monoid is either the trivial group or the cyclic group of order 2. Observe that this result does not obtain for nondeterministic soliton automata of this kind.

The paper is structured as follows: In Section 2 we introduce the basic notions and notation used throughout the paper. In Section 3 we announce and prove our main result.

This paper is self-contained in that every nonstandard notion which is used is also introduced. On the other hand, it does not repeat the background information, the motivation, and the illustrating examples of earlier papers. For these items, the reader is asked to consult the papers listed as references.

2. Basic notions

In this section we review a few basic notions required in the rest of this paper.

The set of positive integers is denoted by $\mathbb{N}$. For the nonnegative integers we use $\mathbb{N}_0$. $\mathbb{Z}$ denotes the integers. An *alphabet* is a finite, nonempty set. Let X be an alphabet. Then X^* denotes the set of *words* over X including the *empty word* ε, and $X^+ = X^* \setminus \{\varepsilon\}$. With the concatenation as multiplication, X^* and X^+ are the free monoid and the free semigroup over X. For a word $w \in X^*$, $|w|$ is the *length* of w; in particular, $|\varepsilon| = 0$. By w^R we denote the *reversal* of w.

A *deterministic finite automaton* is a construct $\mathscr{A} = (S, X, \delta)$ with the following properties: S is a finite, nonempty set, the set of *states*; X is an alphabet, the *input alphabet*; δ is a mapping of $S \times X$ into S, the *transition function*. As nearly all automata considered in this paper are deterministic and finite, we just use the term "automaton" to mean "deterministic finite automaton". Automata without outputs as defined here are also referred to as *semi-automata* in the literature. Occasionally we also need to consider a *nondeterministic automaton*. In this case, the transitions are defined by a mapping δ of $A \times X$ into 2^A instead of into A.

Let $\mathscr{A} = (S, X, \delta)$ be an automaton. As usual, δ is extended to a mapping of $S \times X^*$ into S by

$$\delta(s, \varepsilon) = s \quad \text{and} \quad \delta(s, wx) = \delta(\delta(s, w), x)$$

for $s \in S$, $w \in X^*$ and $x \in X$. For $w \in X^*$ let δ_w denote the transformation of S which is given by

$$\delta_w(s) = \delta(s, w)$$

for $s \in S$. Let

$$T(\mathscr{A}) = \{\delta \mid \delta \in S^S \text{ and } \delta = \delta_w \text{ for some } w \in X^*\}.$$

With the usual multiplication of mappings, the set $T(\mathscr{A})$ is a monoid, the *transition monoid* of $\mathscr{A}$. Clearly, $\delta_u \delta_v = \delta_{uv}$ for any $u, v \in X^*$. Hence, the mapping $X^* \to T(\mathscr{A}) : w \mapsto \delta_w$ is a surjective morphism. To a certain extent, the transition monoid can be used to describe the structure of an automaton and to compare the structure of automata.

For any finite nonempty index set I and for $i \in I$ let $\mathscr{A}_i = (S_i, X_i, \delta_i)$ be an automaton. Their product

$$\prod_{i \in I} \mathscr{A}_i$$

is the automaton $\mathscr{A} = (S, X, \delta)$ where S is the Cartesian product of all S_i, X is the disjoint union of all X_i and

$$\delta((s_i)_{i\in I}, x) = (s_i')_{i\in I}$$

where

$$s_i' = \begin{cases} \delta_i(s_i, x) & \text{if } x \in X_i, \\ s_i & \text{otherwise.} \end{cases}$$

Clearly, $T(\mathscr{A}) \simeq \prod_{i\in I} T(\mathscr{A}_i)$, and the isomorphism is induced by the inclusion mapping of the sets X_i in X.

We also need a few basic notions from classical algebra. The symmetric group on a set of n elements is denoted by $\mathfrak{S}_n$. $\mathbb{Z}_k$ denotes the cyclic group of order k in its natural representation as a factor group of $\mathbb{Z}$.

We now quote the relevant definitions and some results concerning soliton automata from [7]. We start with the necessary notions concerning graphs.

A *graph* is a pair $G = (N, E)$ with N the set of *nodes* and with $E \subseteq N \times N$ the set of *edges*. In this paper we consider only finite, undirected graphs. Therefore, in the sequel we assume without special mention that N is finite and that $E^{-1} \subseteq E$ where $E^{-1} = \{(n, n') \mid (n', n) \in E\}$. Note that with this definition, (n, n') and (n', n) denote the same edge. A mapping $w: N \times N \to \mathbb{N}_0$ is called a *weight function* if

$$w(n, n') = \begin{cases} 0 & \text{for } (n, n') \notin E, \\ w(n', n) > 0 & \text{for } (n, n') \in E. \end{cases}$$

A triple $G = (N, E, w)$ with (N, E) a graph and w a weight function is called a *weighted graph*. For a node $n \in N$ the set

$$V(n) = \{n' \mid (n, n') \in E\}$$

is the *vicinity* of n, the integer

$$d(n) = |V(n)|$$

is its degree, and

$$w(n) = \sum_{n' \in V(n)} w(n, n')$$

is its weight. A node n is said to be *isolated* if $d(n) = 0$, *exterior* if $d(n) = 1$, and *interior* if $d(n) > 1$. A *path* is a word $n_0 n_1 \ldots n_k$ in N^* such that $k > 0$ and $(n_i, n_{i+1}) \in E$ for $i = 0, \ldots, k-1$. The *length* of this path is k. Suppose that $E' \subseteq E$, $N' \subseteq N$, and $w': N' \times N' \to \mathbb{N}_0$. Then N' is the *restriction* of N to E' if N' is the smallest subset of N with $E' \subseteq N' \times N'$. Similarly, w' is the *restriction* of w to E' if $w'(n, n') = w(n, n')$ if $(n, n') \in E'$ and $w'(n, n') = 0$ otherwise.

Definition 2.1 (Dassow, Jürgensen [7]). A *soliton graph* is a weighted graph $G = (N, E, w)$ which satisfies the following conditions:

(a) G has no loops, that is $(n, n) \notin E$ for all $n \in N$.

(b) Every component, that is, maximal connected subgraph, of G has at least one exterior node.

(c) For every $n \in N$ one has $1 \leqslant d(n) \leqslant 3$.
(d) If n is an exterior node then $w(n) \in \{1, 2\}$.
(e) For every $n \in N$ with $d(n) \in \{2, 3\}$ one has $w(n) = d(n) + 1$.

Definition 2.2 (Dassow, Jürgensen [7]). Let $G = (N, E, w)$ be a soliton graph. A path $n_0 \dots n_k$ of G is called a *partial soliton path* if the following conditions hold:
(a) n_0 is an exterior node.
(b) $n_1, \dots, n_{k-1}$ are interior nodes.
(c) There is a sequence $G_0, G_1, \dots, G_k$ of weighted graphs $G_i = (N, E, w_i)$ which can be constructed as follows:
   (c1) $G_0 = G$.
   (c2) For $i = 0, 1, \dots, k-2$, the graph G_{i+1} is defined if and only if G_i is defined and $|w_i(n_i, n_{i+1}) - w_i(n_{i+1}, n_{i+2})| = 1$. In this case

$$w_{i+1}(n, n') = \begin{cases} w_i(n, n') & \text{if } (n, n') \notin \{(n_i, n_{i+1}), (n_{i+1}, n_i)\}, \\ 3 - w_i(n_i, n_{i+1}) & \text{otherwise,} \end{cases}$$

   for all $n, n' \in N$.
   (c3) G_k is defined if and only if G_{k-1} is defined. In this case

$$w_k(n, n') = \begin{cases} w_{k-1}(n, n') & \text{if } (n, n') \notin \{(n_{k-1}, n_k), (n_k, n_{k-1})\}, \\ 3 - w_{k-1}(n, n') & \text{otherwise,} \end{cases}$$

   for all $n, n' \in N$.

Such a partial soliton path is called a *soliton path* if n_k is an exterior node.

Given a soliton graph $G = (N, E, w)$ and a pair of exterior nodes $n, n' \in N$, let $S(G, n, n')$ be the set of weighted graphs which can be obtained as G_k according to the construction given in Definition 2.2 for some soliton path $n_0 \dots n_k$ with $n = n_0$ and $n' = n_k$. Informally, we say that the edge (n_i, n_{i+1}) is *traversed at time i*. We say that G' is *generated by a transition from G*—or G is *transformed into G'*—if and only if $G' \in S(G, n, n')$ for some exterior nodes $n, n' \in N$.

Lemma 2.3 (Dassow, Jürgensen [7]). *Let G be a soliton graph and let $G' \in S(G, n, n')$ for some exterior nodes n, n′ of G. Then G′ is also a soliton graph and $G \in S(G', n, n')$. If p is a soliton path without repeated nodes then also p^R is a soliton path.*

For a soliton graph G, let $S(G)$ denote the set of all soliton graphs which can be generated from G by iterated transitions. This serves as the set of states of a soliton automaton.

Lemma 2.4 (Dassow, Jürgensen [7]). *Let G be a soliton graph and $G' \in S(G)$. Then $S(G) = S(G')$.*

Definition 2.5 (Dassow, Jürgensen [7]). Let G be a soliton graph with X its set of exterior nodes. The *soliton automaton* based on G is defined as

$$\mathscr{A}(G) = (S(G), X \times X, \delta)$$

subject to the following conditions:

(a) $S(G)$ is the set of states.

(b) $X \times X$ is the input alphabet.

(c) $\delta : S(G) \times X \times X \to 2^{S(G)}$ is the nondeterministic transition function with

$$\delta(G', n, n') = \begin{cases} S(G', n, n'), & \text{if } X(G', n, n') \neq \emptyset, \\ \{G'\} & \text{otherwise,} \end{cases}$$

for $G' \in S(G)$ and $n, n' \in X$.

Usually a soliton automaton will have several equivalent input symbols, that is, input symbols which cause exactly the same transitions. In the sequel, equivalent inputs are ignored.

Definition 2.6 (Dassow, Jürgensen [7]). Let G be a soliton graph. G is called *deterministic* if $|S(G', n, n')| \leq 1$ for all $G' \in S(G)$ and all $n, n' \in X$. It is called *strongly deterministic* if for every $G' \in S(G)$ and for every pair of exterior nodes n, n' there is at most one soliton path from n to n' in G'.

If G is a deterministic soliton graph then $\mathscr{A}(G)$ is a deterministic automaton in the usual sense. The soliton automaton $\mathscr{A}(G)$ is said to be *strongly deterministic* if G is strongly deterministic.

In this paper, as the basic tool for expressing the computational power of a soliton automaton $\mathscr{A}(G)$ we consider its transition monoid $T(\mathscr{A}(G))$. We use the convention that the transition monoid of the empty soliton automaton is the singleton monoid. If G is a soliton graph with connected components $G_1, \ldots, G_r$ then

$$T(\mathscr{A}(G)) \simeq \prod_{i=1}^{r} T(\mathscr{A}(G_i)).$$

Therefore, we can restrict our attention to connected graphs G. However, even further restrictions are possible.

Definition 2.7 (Dassow, Jürgensen [7]). Let $G = (N, E, w)$ be a soliton graph. An edge $(n, n') \in E$ is said to be *impervious* if it is not contained in any partial soliton path of any soliton graph $G' \in S(G)$. A path of G is called impervious if each of its edges is.

Impervious edges can always be removed from a soliton graph without affecting its behavior as an automaton [7, 8]. A *reduced* soliton graph is a soliton graph which does not contain any impervious paths. An *indecomposable* soliton graph is a connected, reduced soliton graph. A *chestnut* is a soliton graph consisting of a cycle of even length and some paths entering the cycle subject to the following conditions: Entry points of different paths leading to the cycle have even distances; paths leading to the cycle may meet only at even distances from entry into the cycle.

Theorem 2.8 (Dassow, Jürgensen [7]). *Let $G=(N, E, w)$ be an indecomposable soliton graph. Then G is strongly deterministic if and only if G is a chestnut or (N, E) is a tree. Moreover, if G is strongly deterministic then $T(\mathscr{A}(G))$ is a primitive group of permutations.*

[7, Proposition 5.2 and Corollary 5.3] together imply the following result about deterministic soliton graphs.

Theorem 2.9 (Dassow, Jürgensen [8]). *Let $G=(N, E, w)$ be a deterministic and indecomposable soliton graph. Suppose that (N, E) contains a cycle $p=n_0 \ldots n_k$ with k even, $n_0=n_k$, and $n_j \neq n_l$ for $0 \leq j<l<k$. If there is a $G' \in S(G)$ having a soliton path which contains p as a subpath, then G is a chestnut, hence strongly deterministic, and $T(\mathscr{A}(G)) \simeq \mathfrak{S}_2$.*

In view of Theorem 2.9 one needs to note that an indecomposable deterministic soliton graph can contain a cycle of even length and still not be a chestnut. In this case the cycle would be unusable. An example is provided in [8]. In this context, a cycle is called *usable* if there is a soliton path of which it is a subpath.

For notions not defined in this paper, the reader should consult [13] concerning automata and languages and [7] for soliton automata.

3. The main result

Theorem 3.1. *Let $G=(N, E, w)$ be a deterministic, indecomposable soliton graph with a single exterior node. If G contains a usable cycle of even length then G is a chestnut, $\mathscr{A}(G)$ has two states and $T(\mathscr{A}(G)) \simeq \mathfrak{S}_2$. Otherwise, $\mathscr{A}(G)$ has a single state only and $T(\mathscr{A}(G))$ is trivial.*

Proof. As G has only one exterior node the underlying graph (N, E) contains a cycle. The proof of [7, Proposition 5.4], shows that G contains a usable simple cycle as G is indecomposable. If its length is even then G is a chestnut by [8, Theorem 5.2]. By [7], one has $T(\mathscr{A}(G)) \simeq \mathfrak{S}_2$ in this case.

Now assume that G has no usable simple cycle of even length; hence, it has a usable simple cycle of odd length. We claim that in this case $T(\mathscr{A}(G))$ is trivial. This statement is proved by induction on the number x of nodes of degree 3 of G.

First note that x is always odd. To see this, consider the given graph (N, E) as constructed by the following steps: Start with a graph H_0 consisting of a cycle and a path from the exterior node to that cycle, both without repeated edges. H_0 has one node of degree 3. Assume that H_i has been obtained, that $H_i \neq (N, E)$, and that H_i has an odd number of nodes of degree 3. H_{i+1} is obtained by adding a path of (N, E) without repeated edges which connects two nodes of H_i. These nodes must be distinct and of degree 2. Then H_{i+1} has two additional nodes of degree 3, that is, again, their number is odd.

For the actual induction proof, first consider the case of $x=1$. Then $T(\mathscr{A}(G))$ is trivial by [8, Theorem 6.4].

Now suppose that $x>1$ and that the statement is true for all deterministic indecomposable soliton graphs with a single exterior node, with no usable cycle of even length, and with fewer than x nodes of degree 3. Let p be a soliton path of G. We distinguish two cases; the latter of these has several subcases.

Case (A): *Suppose that G has an edge which is not part of p.* Then G has a path $q=n_0n_1\ldots n_k$ without repeated edges which is not a subpath of p and which satisfies

$$d(n_0)=d(n_k)=3$$

and

$$d(n_1)=d(n_2)=\cdots=d(n_{k-1})=2.$$

We now construct a sequence

$$G_0=G, G_1, G_2, \ldots, G_i, \ldots$$

of weighted graphs such that G_i is obtained from G_{i-1} by the removal of a path q_i whose edges do not occur in p; moreover, G_i has $x_i=x-2i$ nodes of degree 3. The construction process ends when the first G_i, $i>0$, is encountered which is a soliton graph. This is guaranteed to happen as G has a soliton path. Moreover, then p is a soliton path of that final soliton graph G_i. By the induction assumption, $T(\mathscr{A}(G_i))$ is trivial. As only such parts of G have been removed in the construction of G_i which are not part of p, it follows that p causes the identity transformation of G. Determinism then implies that $T(\mathscr{A}(G))$ is indeed also trivial.

To construct G_1 from G_0 we consider two cases. If $n_0\neq n_k$ then let $q_1=q$. If $n_0=n_k$ then there is a path

$$q'=n'_0n'_1\ldots n'_r$$

whose edges do not occur in p and which satisfies

$$n'_0\neq n_k,\qquad d(n'_0)=3,\qquad n'_r=n_0,$$

and

$$d(n'_1)=d(n'_2)=\cdots=d(n'_{r-1})=2.$$

In this case, let $q_1=n'_0n'_1\ldots n'_{r-1}q$. We may assume that q_1 contains no repeated edges. Now let

$$N_1=\begin{cases} N\backslash\{n_1,\ldots,n_{k-1}\} & \text{if } n_0\neq n_k,\\ N\backslash\{n'_1,\ldots,n'_{r-1},n_0,\ldots,n_{k-1}\} & \text{if } n_0=n_k,\end{cases}$$

and

$$E_1=E\cap(N_1\times N_1).$$

Let w_1 be the restriction of w to E_1 and let $G_1=(N_1,E_1,w_1)$. Clearly, the number of nodes of degree 3 has been decreased by 2, that is, $x_1=x-2$.

Now suppose that $G_1, G_2, \ldots, G_i$ have been obtained by the successive removal of paths $q_1, q_2, \ldots, q_i$ which begin and end on a node of degree 3, which have no repeated edges, and whose edges do not occur in p. Moreover, assume that $x_i = x - 2i$. Suppose that $G_i = (N_i, E_i, w_i)$ is not a soliton graph. Then there are nodes $m_1, m_2, m_3 \in N_i$ such that

$$(m_1, m_2), (m_2, m_3) \in E_i, \qquad d(m_2) = 2,$$

and

$$w(m_1, m_2) = w(m_2, m_3) = 1.$$

Therefore, the paths $m_1m_2m_3$ and $m_3m_2m_1$ are not subpaths of p. Moreover, there is a j, $1 \leq j \leq i$, such that m_2 is the first or last node of q_j. As no edge of q_j is in p, it follows that the edges (m_1, m_2) and (m_2, m_3) do not occur in p. Hence, there is a path

$$\bar{q} = \bar{n}_0\bar{n}_1 \ldots \bar{n}_s$$

containing the path $m_1m_2m_3$ and such that

$$d(\bar{n}_0) = d(\bar{n}_s) = 3$$

and

$$d(\bar{n}_1) = d(\bar{n}_2) = \cdots = d(\bar{n}_{s-1}) = 2.$$

We may assume that no edge of $\bar{q}$ occurs in p. If $\bar{n}_0 \neq \bar{n}_s$ then let $q_{i+1} = \bar{q}$. Otherwise, there is a path

$$\bar{q}' = \bar{n}'_0\bar{n}'_1 \ldots \bar{n}'_t$$

whose edges do not occur in p and which satisfies

$$\bar{n}'_0 \neq \bar{n}_s, \qquad d(\bar{n}'_0) = 3, \qquad \bar{n}'_t = \bar{n}_0,$$

and

$$d(\bar{n}'_1) = d(\bar{n}'_2) = \cdots = d(\bar{n}'_{t-1}) = 2.$$

Then let $q_{i+1} = \bar{n}'_0\bar{n}'_1 \ldots \bar{n}'_{t-1}\bar{q}$. Again we may assume that q_{i+1} has no repeated edges. Now define $G_{i+1} = (N_{i+1}, E_{i+1}, w_{i+1})$ by

$$N_{i+1} = \begin{cases} N_i \backslash \{\bar{n}_1, \ldots, \bar{n}_{s-1}\} & \text{if } \bar{n}_0 \neq \bar{n}_s, \\ N_i \backslash \{\bar{n}'_1, \ldots, \bar{n}'_{t-1}, \bar{n}_0, \ldots, \bar{n}_{s-1}\} & \text{if } \bar{n}_0 = \bar{n}_s, \end{cases}$$

and

$$E_{i+1} = E_i \cap (N_{i+1} \times N_{i+1})$$

with w_{i+1} the restriction of w_i to E_{i+1}. It follows that $x_{i+1} = x_i - 2 = x - 2(i+1)$.

This shows that every soliton path which does not include all edges of G induces the identity transformation.

Case (B): *Suppose that every edge of G occurs in the path p, and let p be of minimal length.* To fix notation, let

$$p = n_0n_1 \ldots n_rn_{r+1} \ldots n_sn_{s+1} \ldots n_t \ldots n_u$$

where

$$d(n_0) = 1, \qquad d(n_1) = \cdots = d(n_{r-1}) = 2, \qquad d(n_r) = 3,$$

$$d(n_{r+1}) = \cdots = d(n_{s-1}) = 2, \qquad d(n_s) = 3,$$

$$n_t = n_r, \quad n_r \notin \{n_{r+1}, \ldots, n_{t-1}\}, \quad \text{and}$$

$$n_u = n_0.$$

Note that t exists because n_r is the first node on p at which "branching" could be possible. Moreover, from $x \geq 3$ it follows that s exists and that $n_s \neq n_r$. More precisely, assume that there is no such node n_s. Then the nodes

$$n_r, n_{r+1}, \ldots, n_{t-1}$$

are distinct and form a usable cycle which is of odd length by assumption. Hence, the path

$$\bar{p} = n_0 \ldots n_{r-1} n_r n_{r+1} \ldots n_{t-1} n_t n_{r+1} \ldots n_{t-1} n_t n_{r-1} \ldots n_0$$

is a soliton path. If $\bar{p} = p$ then, by the indecomposability of G, one has $x = 1$, a contradiction! On the other hand, if $\bar{p} \neq p$ then determinism implies that p and $\bar{p}$ induce the same transformation, that is, the identity transformation.

Now let $\bar{n}_{r+1}$ be the node with $(n_r, \bar{n}_{r+1}) \in E$ and $\bar{n}_{r+1} \notin \{n_{r-1}, n_{r+1}\}$. We distinguish a few subcases. In these cases we typically use the following argument: From p we construct a soliton path $\bar{p}$ which misses at least one edge. By case (A), $\bar{p}$ induces the identity transformation. Determinism implies that also p induces the identity transformation.

Case (B1): *Suppose that* $w(n_{r-1}, n_r) = 1$ *and* $n_{t-1} = n_{r+1}$. Then $w(n_r, n_{r+1}) = 2$ and $w(n_r, \bar{n}_{r+1}) = 1$. The undirected edge $(n_r, \bar{n}_{r+1})$ is not part of the segment

$$n_0 n_1 \ldots n_r n_{r+1} \ldots n_s n_{s+1} \ldots n_t$$

of p because of the choice of t. After the first passage through n_{r+1} the weights of the edges (n_{r-1}, n_r) and (n_r, n_{r+1}) are 2 and 1 respectively. Therefore,

$$n_{t+1} = n_{r-1}, \ldots, n_{u-1} = n_1, n_u = n_0.$$

This implies that the edge $(n_r, \bar{n}_{r+1})$ does not occur in p at all, a contradiction! Therefore, case (B1) is impossible.

Case (B2): *Suppose that* $w(n_{r-1}, n_r) = 2$ *and* $n_{t-1} = n_{r+1}$. Then $w(n_r, n_{r+1}) = w(n_r, \bar{n}_{r+1}) = 1$. As in case (B1), the edge $(n_r, \bar{n}_{r+1})$ does not occur in the first part of p up to n_t. Then the path

$$\bar{p} = n_0 \ldots n_r n_{r+1} \ldots n_{t-1} n_t n_{r-1} \ldots n_0$$

is a soliton path not containing the edge $(n_r, \bar{n}_{r+1})$. Hence p induces the identity transformation.

Case (B3): *Suppose that* $w(n_{r-1}, n_r) = 2$ *and* $n_{t-1} = \bar{n}_{r+1}$. Then $w(n_r, n_{r+1}) = w(n_r, \bar{n}_{r+1}) = 1$ and $n_{t+1} = n_{r+1}$. Note that the undirected edges (n_r, n_{r+1}) and $(n_r, \bar{n}_{r+1})$ occur exactly once in the initial segment of p up to n_t. Consider the situation at n_s and let $\bar{n}_{s+1}$ be the node with $(n_s, \bar{n}_{s+1}) \in E$ and $\bar{n}_{s+1} \notin \{n_{s-1}, n_{s+1}\}$.

Suppose that $w(n_s, n_{s+1}) = 2$, that is, $w(n_{s-1}, n_s) = w(n_s, \bar{n}_{s+1}) = 1$.

Assume that n_s occurs more than once in the path $n_r \ldots n_t$. Then

$$\bar{p} = n_0 \ldots n_{r-1} n_r \ldots n_{s-1} n_s n_{s+1} \ldots n_s n_{s-1} \ldots n_{r+1} n_r n_{r-1} \ldots n_0$$

is a soliton path by the assumptions about r, s and t. This path does not contain the edge $(n_r, \bar{n}_{r+1})$. Hence p induces the identify transformation.

On the other hand, assume that n_s occurs only once in the path $n_r \ldots n_t$. Then the path

$$\bar{p} = n_0 \ldots n_{r-1}n_rn_{r+1} \ldots n_{s-1}n_sn_{s+1} \ldots n_{t-1}n_tn_{r+1} \ldots n_{s-1}n_sn_{s+1} \ldots n_{t-1}n_tn_{r-1} \ldots n_0$$

is a soliton path which does not contain the edge $(n_s, \bar{n}_{s+1})$. Again this implies that p induces the identity transformation.

Now suppose that $w(n_{s-1}, n_s) = 2$. Then

$$w(n_s, n_{s+1}) = w(n_s, \bar{n}_{s+1}) = 1.$$

If

$$n_s \notin \{n_{s+1}, \ldots, n_{t-1}\}$$

then, in addition to p, there is the soliton path

$$\bar{p} = n_0 \ldots n_rn_{r+1} \ldots n_sn_{s+1} \ldots n_rn_{r+1} \ldots n_sn_{s+1} \ldots n_r \ldots n_0$$

which does not contain the edge $(n_s, \bar{n}_{s+1})$. As above it follows that p causes the identity transformation. Therefore, assume that

$$n_s \in \{n_{s+1}, \ldots, n_{t-1}\},$$

say, $n_s = n_v$ with $s+1 < v \leq t-1$. Let v be minimal with this property. At the time when n_v is about to be passed, the weights of (n_{s-1}, n_s), (n_s, n_{s+1}), and $(n_s, \bar{n}_{s+1})$ are 1, 2, and 1, respectively.

If $n_{v-1} = n_{s+1}$ then, in addition to p, there is a soliton path

$$\bar{p} = n_0 \ldots n_r \ldots n_sn_{s+1} \ldots n_{v-1}n_vn_{s-1} \ldots n_rn_{r-1} \ldots n_0$$

which does not contain the edge $(n_s, \bar{n}_{s+1})$. As before this implies that p induces the identity.

It is impossible that $n_{v-1} = n_{s-1}$. Therefore, the only remaining possibility is that $n_{v-1} = \bar{n}_{s+1}$. In this case $n_{v+1} = n_{s+1}$. Let w be maximal with $w > v$ such that $n_{v+j} = n_{s+j}$ for $j = 1, \ldots, w$. Because of the minimality of p, $s + w < v$. Then

$$\bar{p} = n_0 \ldots n_r \ldots n_s\bar{n}_{s+1} \ldots n^{v+w+1}n_{v+w}n_s + w + 1 \ldots n_r \ldots n_0$$

or

$$\bar{\bar{p}} = n_0 \ldots n_{r-1}n_rn_{r+1} \ldots n_{s-1}n_s\bar{n}_{s+1} \ldots n_{s+w} \ldots n_s \ldots n_{s+w} \ldots n_sn_{s-1} \ldots n_r \ldots n_0$$

is a soliton path. The former misses the edge (n_s, n_{s+1}); the latter misses $(n_r, \bar{n}_{r+1})$. Hence, p induces the identity.

Case (B4): *Suppose that* $w(n_{r-1}, n_r) = 1$ *and* $n_{t-1} = \bar{n}_{r+1}$. This case is analogous to case (B3). □

In view of the general goal of characterizing deterministic soliton automata, the result of Theorem 3.1 provides a useful necessary condition.

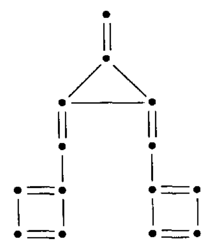

Fig. 1. A nondeterministic soliton graph with a single exterior node and four states.

Corollary 3.2. *Let G be a deterministic indecomposable soliton automaton which is not a chestnut. Then, for every exterior node n, the input (n, n) induces the identity transformation.*

The results do not obtain without the assumption of determinism. An example is provided in Fig. 1.

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